

The flow caused by the differential rotation of a right circular cylindrical depression in one of two rapidly rotating parallel planes

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(Received 20 October 1971)

The flow induced by the differential rotation of a cylindrical depression of radius a in one of two parallel rigid planes rapidly rotating about their common normal at speed Ω is studied. A Taylor column bounded by the usual Stewartson layers arises, but the shear-layer structure is rather different from any previously studied. The $E^{\frac{1}{2}}$ -layers ($E \equiv \nu/\Omega a^2$) smooth the discontinuity in the geostrophic flow, but the way in which this is accomplished is related to the possible singularities of the $E^{\frac{1}{2}}$ -layer solutions. The fact that the $\frac{1}{3}$ -layer is partially free and partially attached to a vertical boundary accounts for the new joining conditions for the $\frac{1}{4}$ -layer. The drag on a right circular cylindrical bump in uniform flow is given in addition to some general comments on the applicability of these joining conditions to the motion of an axisymmetric object of quite general shape.

1. Introduction

In recent years, several investigators have presented solutions to the Navier-Stokes equations for motion in rigidly rotating incompressible fluids. An important feature of many of the problems, owing to the existence of a Taylor column, is the shear-layer structure laterally bounding such columns, first given careful study by Stewartson (1957). Such shear layers may be free, as in the work of Stewartson (1966) and Moore & Saffman (1969*a*), or attached to a solid vertical boundary; they may in some circumstances exert some degree of control over the geostrophic flow (Moore & Saffman 1969*b*).

The geometrical configuration of the problem studied here is given in figure 1. In one of two rapidly rotating parallel plates separated by a distance ah there is a right circular cylindrical depression of radius a and depth ad . Let the base of that cylinder be denoted by R_1 and the cylinder of fluid above it by D_1 ; let the part of the lateral boundary of that cylinder coincident with the solid boundary be R_3 . D_2 is the remainder of the domain exterior to D_1 and is bounded beneath by R_2 . We shall use non-dimensional cylindrical polar co-ordinates (r, θ, z) throughout, with the associated velocity components (u, v, w) . We shall consider the flow induced when R_1 rotates at speed $\Omega(1 + \epsilon)$ and R_3 at speed $\Omega(1 + \epsilon')$ in the limit $E \rightarrow 0$, with ϵ and ϵ' small; the angular rotation rate of the remainder of the container is Ω .

In §2, we give the solutions in D_1 and D_2 , which exhibit the characteristic

discontinuities on $r = 1$. The smoothing of these discontinuities by the Stewartson shear layers is discussed in §3. These layers are quite different from either of those studied by Stewartson (1966) or Moore & Saffman (1969*a*) because they are free shear layers for $z > 0$, but fixed on the boundary R_3 for $z < 0$. The structure of these $\frac{1}{3}$ -layer solutions and, hence, the matching conditions for the $\frac{1}{4}$ -layers, which are rather different from any relating to the problems detailed by Moore & Saffman (1969*a*), are given.

The solutions in §3 are difficult since, as has been known for some time, the $\frac{1}{3}$ -layer problem is not well posed, although some specification of possible singularities in the domain can be given. In §4, a technique first used by Moore & Saffman (1969*a*) is applied to elucidate the kinds of singularities possible near a convex (270°) corner. The results given in §3 do not degenerate to those of Stewartson (1957) when $\delta \equiv d/h$ is small. In §4, it is indicated that only so long as $\delta \ll E^{\frac{1}{2}}$ will Stewartson's result be recovered. A solution to the $\frac{1}{3}$ -layer problem is given at the end of §3 and is valid for $E^{\frac{1}{2}} \ll \delta \ll 1$.

Once the above problem has been understood, it requires only a small extension to deal with rectilinear flow past a cylindrical bump of height ad . The drag is found to be $O(\mu U a d h^{-\frac{1}{2}} E^{-\frac{1}{4}})$. Finally, §6 contains some comments on the proper procedure for joining $\frac{1}{4}$ -layers for an object of quite general shape.

Geometrically related configurations to those above have been studied experimentally (Taylor 1923; Hide & Ibbetson 1966; Hide & Titman 1967), but, whereas qualitative agreement is easy to see, quantitative comparison is not possible since the parameter range in the experiments is not in accord with the restrictions on the analysis which are $\max(\epsilon, \epsilon') \ll E^{\frac{1}{2}} h^{\frac{1}{2}}, d/h \gg E^{\frac{1}{2}}$.

2. Formulation and interior solution

The equations of motion for a fluid in a co-ordinate system rotating at speed Ω are

$$\nabla \cdot \mathbf{u} = 0, \quad (2.1a)$$

$$D\mathbf{u}/Dt + 2\mathbf{k} \times \mathbf{u} + \nabla p = E\nabla^2 \mathbf{u}, \quad (2.1b)$$

where velocities have been made non-dimensional with Ωa , lengths with a , and $\mathbf{k} = \boldsymbol{\Omega}/\Omega$. As was described in §1, we seek a solution to (2.1) subject to the boundary conditions

$$\mathbf{u} = \epsilon \mathbf{k} \times \mathbf{r} \quad \text{on } R_1, \quad \mathbf{u} = 0 \quad \text{on } R_2, \quad (2.2a, b)$$

$$\mathbf{u} = \epsilon' \mathbf{k} \times \mathbf{r} \quad \text{on } R_3, \quad \mathbf{u} = 0 \quad \text{on } z = h, \quad (2.2c, d)$$

so that \mathbf{u} is actually of order $\max(\epsilon, \epsilon')$. If ϵ and ϵ' are sufficiently small, as stated in §1, then the first term of (2.1*b*) will be small compared with the second and hence negligible. For purposes of this analysis then, we replace (2.1) by

$$\left. \begin{aligned} \nabla \cdot \mathbf{u} &= 0, \\ 2\mathbf{k} \times \mathbf{u} + \nabla p &= E\nabla^2 \mathbf{u}. \end{aligned} \right\} \quad (2.3)$$

An essential feature in the construction of solutions to (2.3) with (2.2) as $E \rightarrow 0$, even in the thin vertical shear layers (Jacobs 1964), is the compatibility

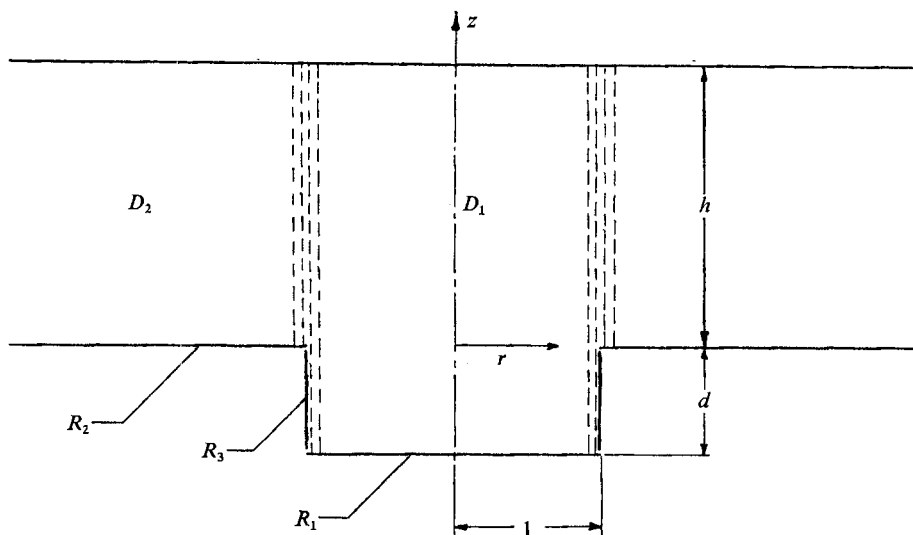


FIGURE 1. The geometry of the problem. Lengths have been made dimensionless with a , the radius of the surface R_1 ; broken lines represent the shear layers. D_1 and D_2 are the regions interior and exterior to $r = 1$ respectively.

condition on the velocity components imposed by the structure of the Ekman layer on horizontal surfaces:

$$w = \frac{1}{2}(\mathbf{n} \cdot \mathbf{k}) E^{\frac{1}{2}} \left[\frac{1}{r} \frac{\partial}{\partial r} (r(v - v_B)) - \frac{1}{r} \frac{\partial u}{\partial \theta} \right], \tag{2.4}$$

where \mathbf{n} is the outward normal to the horizontal surface and v_B denotes the swirl velocity of the boundary. This constraint must, therefore, be satisfied on R_1, R_2 and $z = h$, so long as horizontal rates of change are $o(E^{-\frac{1}{2}})$.

The usual expansion of (\mathbf{u}, p) in rational powers of E and substitution in (2.2)–(2.4) gives the solutions

$$u = 0, \quad v = \frac{1}{2}\epsilon r, \quad w = -\frac{1}{2}\epsilon E^{\frac{1}{2}} \quad \text{in } D_1$$

and

$$u = v = w = 0 \quad \text{in } D_2.$$

These solutions were given first by Stewartson (1957). As one might have expected, the interior solutions are independent of ϵ' , the effects of this parameter being confined to the vertical shear layers.

3. The vertical shear layers

The solutions valid in D_1 and D_2 given in §2 exhibit the now familiar feature of a Taylor column, and its associated discontinuities on $r = 1$; such discontinuities are to be smoothed by Stewartson layers of thicknesses $E^{\frac{1}{2}}$ and $E^{\frac{1}{2}}$.

If we use upper case letters for the layers of thickness $E^{\frac{1}{2}}$, then the equations valid in that layer, which are geostrophic apart from the retention of azimuthal shear, are

$$\frac{\partial U}{\partial x} + \frac{\partial W}{\partial z} = 0, \quad -2V + \frac{\partial P}{\partial x} = 0, \quad 2U = E \frac{\partial^2 V}{\partial x^2}, \quad \frac{\partial P}{\partial z} = 0, \tag{3.1}$$

where $x \equiv r - 1$. From these equations one finds that

$$\frac{\partial U}{\partial z} = \frac{\partial V}{\partial z} = \frac{\partial^2 W}{\partial z^2} = 0. \tag{3.2}$$

The Ekman conditions (2.4) for the layer on $r = 1^+$ take the form

$$W = \begin{cases} -\frac{1}{2}E^{\frac{1}{2}}\partial V/\partial x & \text{on } z = h, \\ \frac{1}{2}E^{\frac{1}{2}}\partial V/\partial x & \text{on } z = 0, \end{cases}$$

which when combined with (3.1) and (3.2) give

$$\frac{1}{2}E^{\frac{1}{2}}h \frac{d^3 V}{dx^3} - \frac{dV}{dx} = 0. \tag{3.3}$$

The solution of (3.3) which matches the solution in D_2 from §2 is

$$V = A e^{-px}, \quad p^2 = 2/hE^{\frac{1}{2}}. \tag{3.4}$$

The layer on $r = 1^-$ has the solution, valid to $O(E^{\frac{1}{2}})$,

$$V = \frac{1}{2}\epsilon + B e^{qx}, \quad q^2 = 2/h_0 E^{\frac{1}{2}}, \tag{3.5}$$

where $h_0 \equiv h + d$ and A and B are constants not yet determined.

Stewartson (1966) has shown how to determine A and B by use of the equations for the $E^{\frac{1}{2}}$ -layer interior to the two $E^{\frac{1}{2}}$ -layers; Moore & Saffman (1969*a*) have used this technique to study the flow induced by a rising disk. Stewartson (1957) found that for $d \equiv 0$ the correct conditions for determining A and B are that V and dV/dx should be continuous on $x = 0$. Moore & Saffman (1970) found that the jump conditions on the (three) $\frac{1}{4}$ -layers for the rising disk are continuous velocity and zero total shear stress on the column, the latter of which means that dV/dx is discontinuous at $x = 0$. Neither of these jump conditions is appropriate here because of the no-slip condition on R_3 .

The equations valid in the shear layer of width $E^{\frac{1}{2}}$ are

$$\frac{\partial^3 \tilde{v}}{\partial \eta^3} = -2 \frac{\partial \tilde{w}}{\partial z}, \quad \frac{\partial^3 \tilde{w}}{\partial \eta^3} = 2 \frac{\partial \tilde{v}}{\partial z}, \tag{3.6}$$

where $\eta \equiv x/E^{\frac{1}{2}}$ and a tilda denotes a function of η and z . For η large, \tilde{v} and \tilde{w} must match the $\frac{1}{4}$ -layer solutions given previously. Assume that the $\frac{1}{3}$ -layer expansions begin as

$$\begin{aligned} \tilde{v} &= v_0 + E^{\frac{1}{3}}v_1 + E^{\frac{2}{3}}v_2 + \dots, \\ \tilde{w} &= w_0 + E^{\frac{1}{3}}w_1 + E^{\frac{2}{3}}w_2 + \dots \end{aligned}$$

Now, (v_0, w_0) , (v_1, w_1) and (v_2, w_2) each independently satisfy (3.6), so putting $x = E^{\frac{1}{2}}\eta$ into (3.4) and (3.5) and expanding for $E \rightarrow 0$, with η fixed, gives the matching conditions

$$\left. \begin{aligned} v_0 &\sim A & \text{as } \eta \rightarrow +\infty, \\ v_0 &\sim \frac{1}{2}\epsilon + B & \text{as } \eta \rightarrow -\infty, \end{aligned} \right\} \tag{3.7a}$$

$$\left. \begin{aligned} v_1 &\sim -(2/h)^{\frac{1}{2}}A\eta & \text{as } \eta \rightarrow +\infty, \\ v_1 &\sim (2/h_0)^{\frac{1}{2}}B\eta & \text{as } \eta \rightarrow -\infty \end{aligned} \right\} \tag{3.7b}$$

for the first two terms of the asymptotic solution; here A and B are taken to be $O(1)$. Expanding the boundary condition (2.2c) in similar fashion gives

$$v_0 = \epsilon', \quad w_0 = \partial^2 v_0 / \partial \eta^2 = 0, \tag{3.8a}$$

$$v_1 = w_1 = \partial^2 v_1 / \partial \eta^2 = 0 \tag{3.8b}$$

on $\eta = 0$, $-d < z < 0$. Boundary conditions on R_1 , R_2 and $z = h$ are found by writing the Ekman condition (2.4) in terms of $\frac{1}{3}$ -layer variables and expanding in powers of $E^{1/3}$; the result is

$$w_0 = w_1 = 0 \text{ on horizontal boundaries.} \tag{3.9}$$

We now consider the (v_0, w_0) and (v_1, w_1) problems in order to find A and B .

The (v_0, w_0) problem

This problem is the solution of

$$\frac{\partial^3 v_0}{\partial \eta^3} = -2 \frac{\partial w_0}{\partial z}, \quad \frac{\partial^3 w_0}{\partial \eta^3} = 2 \frac{\partial v_0}{\partial z}, \tag{3.10a, b}$$

subject to (3.7a), (3.8a) and (3.9a). Integrating (3.10a) from $z = 0$ to h and using (3.9a) yields

$$\frac{d^3}{d\eta^3} \int_0^h v_0 dz = 0 \quad \text{for } \eta > 0.$$

Similar integration from $z = -d$ to h gives

$$\frac{d^3}{d\eta^3} \int_{-d}^h v_0 dz = 0 \quad \text{for } \eta < 0.$$

By integrating again, with respect to η , with reference to the matching conditions (3.7a), we obtain

$$\int_0^h v_0 dz = Ah \quad \text{for } \eta > 0,$$

$$\int_{-d}^h v_0 dz = (\frac{1}{2}\epsilon + B) h_0 \quad \text{for } \eta < 0.$$

Now, on $\eta = 0^-$

$$\int_{-d}^h v_0 dz = \int_{-d}^0 v_0 dz + \int_0^h v_0 dz = (\frac{1}{2}\epsilon + B) h_0,$$

but
$$\int_{-d}^0 v_0 dz = \epsilon' d$$

by (3.8a), so
$$\epsilon' d + \int_0^h v_0 dz = (\frac{1}{2}\epsilon + B) h_0.$$

However, on $\eta = 0^+$
$$\int_0^h v_0 dz = Ah$$

and since v_0 must be continuous on $\eta = 0$, $0 < z < h$, we have

$$\epsilon' d + Ah = (\frac{1}{2}\epsilon + B) h_0, \tag{3.11}$$

which is a single equation relating A and B . Hocking (1967), in a similar problem, proved that the only non-singular solution to the (v_0, w_0) problem is $v_0 = \text{constant}$.

There is no reason *a priori* why (v_0, w_0) should be non-singular. However, in §4 we show that the minimum singularity hypothesis requires that (v_0, w_0) should have no singularity at $\eta = z = 0$. Hence

$$v_0 = \epsilon', \quad w_0 = 0 \tag{3.12}$$

is the solution. Then, the matching conditions (3.7a) give

$$A = \frac{1}{2}\epsilon + B = \epsilon', \tag{3.13}$$

which makes (3.11) an identity.

The (v_1, w_1) problem

Here, we seek a solution to

$$\frac{\partial^3 v_1}{\partial \eta^3} = -2 \frac{\partial w_1}{\partial z}, \quad \frac{\partial^3 w_1}{\partial \eta^3} = 2 \frac{\partial v_1}{\partial z} \tag{3.14}$$

subject to (3.7b), (3.8b) and (3.9b). Integration of the equation as was done for (v_0, w_0) gives just

$$\int_{-a}^0 \frac{\partial v_1}{\partial \eta} \Big|_{\eta=0} dz = 2h_0 B + 2hA,$$

which imposes no additional constraints on A, B . A simple solution, which seems to be the only non-singular one, is $v_1 = c\eta, w_1 = 0$. Matching by use of (3.7b), however, does impose an additional restriction on A and B . However, in §4 we show that w_1 behaves like $|\eta|^{-\frac{1}{2}}$ as $\eta \rightarrow 0$ on $z = 0$; so $v_1 = c\eta$ is unacceptable and there is thus no contradiction with (3.13).

We note that determination of A and B rests on whether v_0 or v_1 is singular; regularity of both functions at $z = \eta = 0$ imposes three conditions on A and B . Hence, either v_0 or v_1 , or both, must have a singularity. It is only the analysis of §4 that makes possible the choice v_0 regular, v_1 singular.

Reasons are given in §4 for expecting that these jump conditions and a singular solution for (v_1, w_1) are valid for all $d/h \gg E^{\frac{1}{2}}$. If $d/h \ll 1$ but still large compared with $E^{\frac{1}{2}}$, the v_1 solution is easily obtained by noting from (3.14) that since

$$\frac{d^3}{d\eta^3} \int_0^h v_1 dz = 2w_1(\eta, 0),$$

a generalized function boundary condition

$$w_1(\eta, 0) = (\frac{1}{2}h)^{\frac{1}{2}}(\epsilon - 4\epsilon') \delta'(\eta)$$

will satisfy (3.7b) and (3.13), and is hence equivalent to the $|\eta|^{-\frac{1}{2}}$ singularity mentioned above. Equations (3.14) are then easily solved in $|\eta| < \infty, 0 < z < h$, and to leading order in $d/h \ll 1$ we have

$$w_1 = \frac{1}{2\pi i} \left(\frac{h}{2}\right)^{\frac{1}{2}} (\epsilon - 4\epsilon') \int_{-\infty}^{\infty} e^{ik\eta} \frac{\sinh [\frac{1}{2}k^3(z-h)]}{\sinh \frac{1}{2}k^3 h} k dk,$$

which may easily be evaluated by the method of residues so long as $z > 0$. The motion given here is essentially that due to a doublet at $\eta = z = 0$. Solution of the (v_1, w_1) problem for $d = O(h)$ has so far proved too difficult, owing to the peculiar shape of the domain.

4. The singularities of the $\frac{1}{3}$ -layer equations at a convex corner

We seek solutions to the $\frac{1}{3}$ -layer equations (3.6) in the domain

$$z > 0, -\infty < \eta < \infty; \quad z < 0, -\infty < \eta < 0.$$

They may be combined into a single equation:

$$\partial^3 \chi / \partial \eta^3 = -2i \partial \chi / \partial z, \quad \chi \equiv w + iv,$$

as was done by Moore & Saffman (1969*a*). In §3 of their paper similar solutions are given. With $\tau \equiv \eta(2/z)^{\frac{1}{3}}$, they show that, provided $m < 0$,

$$\chi = Az^m H_m(\tau) \quad \text{for } z > 0, \tag{4.1}$$

$$H_m(\tau) \equiv \int_0^\infty e^{-i p \tau} p^{-3m-1} e^{-p^3} dp. \tag{4.2}$$

In $z < 0$ it is convenient to use $\tau' = \eta(2/-z)^{\frac{1}{3}}$, so that $\tau' < 0$ in the domain of interest. Since they are interested only in doubly infinite regions, Moore & Saffman do not write down the solution valid in $z < 0, \eta < 0$, but all the elements are there and, omitting the algebraic details, we have

$$\chi = (-z)^m [B \bar{H}_m(\tau') + C \bar{H}_m(e^{-\frac{1}{3} 2\pi i \tau'})] \quad \text{for } z < 0, \tag{4.3}$$

where an overbar denotes a complex conjugate. Now, the boundary conditions are all homogeneous:

$$\left. \begin{aligned} w = 0 \quad \text{on } z = 0, \eta > 0, \\ v = \partial^2 v / \partial \eta^2 = w = 0 \quad \text{on } \eta = 0, z < 0, \\ v, w \text{ continuous on } z = 0, \eta < 0. \end{aligned} \right\} \tag{4.4}$$

From (4.3), clearly $\chi = 0$ on $\eta = 0, z < 0$ if $B + C = 0$. Thus (4.3) becomes

$$\chi = (-z)^m B [\bar{H}_m(\tau') - \bar{H}_m(e^{-\frac{2}{3} \pi i \tau'})] \quad \text{for } z < 0. \tag{4.5}$$

Now, on $\eta = 0, z < 0$,

$$\partial^2 v / \partial \eta^2 = (-z)^{m-\frac{2}{3}} |B| 2^{\frac{2}{3}} \frac{1}{3} \Gamma(\frac{2}{3} - m) \sin \frac{2}{3} \pi \cos(\beta - \frac{2}{3} \pi),$$

where $B = |B| e^{i\beta}$. Thus

$$\beta = \frac{2}{3} \pi + (k + \frac{1}{2}) \pi, \quad k \text{ an integer.} \tag{4.6}$$

The condition that $w = 0$ on $z = 0, \eta > 0$ gives, as in Moore & Saffman (1969*a*),

$$\alpha = -\frac{3}{2} m \pi + (j + \frac{1}{2}) \pi, \quad j \text{ an integer,} \tag{4.7}$$

with $A = |A| e^{i\alpha}$, and continuity on $z = 0, \eta < 0$ gives

$$\alpha - \frac{3}{2} m \pi = \frac{1}{2} \pi + \beta + \frac{1}{2} m \pi + 2n \pi. \tag{4.8}$$

Now, (4.6)-(4.8) may easily be combined to give

$$m = -\frac{1}{3} - \frac{2}{7} l, \quad (l = -1, 0, 1, 2). \tag{4.9}$$

These singularities are associated with the interaction of the $\frac{1}{3}$ -layer and the Ekman layer near $\eta = z = 0$. By using the arguments given by Moore & Saffman,

	$m = -\frac{1}{2\frac{1}{1}}$	$m = -\frac{1}{3}$	$m = -\frac{1\frac{3}{1}}{2\frac{1}{1}}$
$n = 0$	$E^{-\frac{1}{2\frac{1}{1}}}$	$E^{-\frac{1}{3}}$	$E^{-\frac{1\frac{3}{1}}{2\frac{1}{1}}}$
$n = 1$	$E^{\frac{5}{4}}$	$E^{-\frac{1}{2}}$	$E^{-\frac{1\frac{6}{1}}{2\frac{1}{1}}}$
$n = 2$	$E^{\frac{1}{2}}$	1	$E^{-\frac{1}{2}}$

TABLE 1. The swirl velocity associated with each singularity

we see that, near where it joins to the Ekman layer, the $\frac{1}{3}$ -layer solution $E^{\frac{1}{3}n}$ will have size $E^{\frac{1}{3}n + \frac{1}{2}m}$. Hence, we can construct table 1, which gives the size of the swirl velocity associated with each singularity.

We note here that the complete expansion of the $\frac{1}{3}$ -layer solutions should not be as given in §3, in powers of $E^{\frac{1}{3}}$, but rather in powers of $E^{\frac{1}{3\frac{1}{1}}}$. Such a complication is unnecessary in this context since, though a singularity may arise first in the $E^{\frac{5}{4}}$ term, the matching conditions found from (3.4) and (3.5) make all such solutions zero except the ones corresponding to $E^{\frac{1}{3\frac{1}{1}}}$, $E^{\frac{1\frac{3}{1}}{2\frac{1}{1}}}$, ...

According to Moore & Saffman's principle of minimum singularity, the swirl velocity should be no greater in the shear layers than in the geostrophic flow. Hence, only singularities that give swirl velocities of $O(1)$ or smaller are acceptable. Therefore the $n = 0$ solution must be regular, the $n = 1$ solution must have an $m = -\frac{1}{2\frac{1}{1}}$ singularity, the $n = 2$ solution must contain both $m = -\frac{1}{2\frac{1}{1}}$ and $m = -\frac{1}{3}$ singularities, etc.

If $d \ll h$, then the appropriate z scale near the corner is $\delta \equiv d/h$. The $\frac{1}{3}$ -layer equations are invariant under the transformation $(\eta, z) \leftrightarrow (\eta/\delta^{\frac{1}{3}}, z/\delta)$, so the effect of having $\delta \ll 1$ is to multiply each element in table 1 by δ^m . A little examination shows that the $n = 1$ problem will always be singular if $\delta \gg E^{\frac{1}{2}}$; certainly the analysis can in any case be valid only for $\delta \gg E^{\frac{1}{2}}$ (the requirement that the depression should be deeper than an Ekman-layer thickness). Thus the evaluation of A and B given in §3 appears correct for all $\delta \gg E^{\frac{1}{2}}$. Obviously, when $\delta \ll E^{\frac{1}{2}}$ the $\frac{1}{3}$ -layer 'sees' a flat-bottom boundary and Stewartson's (1957) jump conditions for the $\frac{1}{4}$ -layer in such a geometry are recovered.

5. Flow past a right circular cylinder

Suppose fluid flows at speed $U (= \epsilon\Omega a)$ past a right circular cylinder of height ad resting on one of two parallel planes situated a distance ah apart. The fluid over the cylinder is stagnant (if d is not too small; see Moore & Saffman (1969b)), as it is over a hemisphere in rectilinear flow (Jacobs 1964).

The geostrophic flow in $r > 1$ is given by an harmonic function

$$p = -2\epsilon \sin \theta (r - r^{-1}),$$

and the speed along $r = 1$ is $2\epsilon \sin \theta$. The $\frac{1}{4}$ -layer solutions are

$$V = \begin{cases} 2\epsilon \sin \theta + A e^{-px} & \text{for } r > 1, \\ B e^{qx} & \text{for } r < 1, \end{cases}$$

where all of the symbols have the same meaning as in §3 except that $h_0 \equiv h - d$

here. Using the jump condition (3.13) clearly leads to $B \equiv 0$, i.e. there is no $\frac{1}{4}$ -layer in $r < 1$. Further,

$$2\epsilon \sin \theta + A = 0,$$

hence

$$v = 2\epsilon \sin \theta (1 - e^{-\rho x}) \quad \text{for } r > 1.$$

The drag arises from shear stress on the side walls and is given by

$$D = \pi \mu U a d (2/h)^{\frac{1}{2}} E^{-\frac{1}{2}}.$$

6. Final remarks

We have seen that, if the $\frac{1}{3}$ -layer solution is expanded as

$$v = v_0 + E^{-\frac{1}{2}} v_1 + E^{\frac{1}{2}} v_2 + \dots,$$

a weak singularity $v_1 \sim |\eta|^{-\frac{1}{2}}$ for $\eta \rightarrow 0$ on $z = 0$ arises in the v_1 term if the $\frac{1}{3}$ -layer contains a 270° convex corner. Further, jump conditions for $\frac{1}{4}$ -layers containing such a $\frac{1}{3}$ -layer may be entirely determined from the v_0 solution, which is regular. This result stands in contrast to the results for a 360° corner (Moore & Saffman 1969*a*), where, though v_1 is singular there too, consideration of the v_1 solution is necessary to deduce a second jump condition.

Thus it can be quite generally stated that for the motion through a rotating fluid of an axisymmetric object whose shape is given by $z = \pm f(r)$, with $f(1) = 0$, whether $f'(1)$ is $O(1)$ or infinite, the $\frac{1}{4}$ -layers (or quasi-geostrophic layers of whatever thickness) are joined by requiring $V(x)$ to be continuous and zero net shear stress on the column, as was determined by Moore & Saffman (1969*a*). However, if $f(1) \neq 0$ and there exists an $O(1)$ region on $r = 1$ along which $f'(1) = \infty$, then the quasi-geostrophic layers are to be joined by requiring $V(x)$ to be continuous and also $V(0) = v_B(1)$, where $v_B(r)$ is the swirl velocity of the object.

The author gratefully acknowledges the suggestion of this problem by Professor K. Stewartson and his continuing interest in it during the course of this study. Thanks are also due to Professor Burggraf for his interest and several suggestions.

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